Supplementary Appendix

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1 Additional empirical results

1.1 Smoothness restriction on type-specific time fixed-effects

When the given dataset has relatively small number of units and/or time periods, carefully chosen smoothness restriction on the type-specific time fixed-effect can hugely improve the classification result, as shown in the simulations (see Section 6 of the main text). When there is no *a priori* knowledge on the smoothness restrictions, we suggest using cross-validated mean-square forecasting error, as a selection criterion on the smoothness restriction. For the main specification of the paper, we considered four smoothness restrictions:

$$
\begin{aligned}
&\delta_t(k) = \delta(k), \quad \cdots \textit{constant}\\
&\delta_t(k) = \Big(1, t-1989\Big)\delta(k), \quad \cdots \textit{linear}\\
&\delta_t(k) = \Big(1, t-1989, (t-1993)\mathbf{1}\{t \geq 1993\}\Big)\delta(k), \quad \cdots \textit{linear with a break}\\
&\delta_t(k) = \Big(1, t-1989, (t-1992)\mathbf{1}\{t \geq 1992\},\\
&\qquad\qquad\qquad (t-1995)\mathbf{1}\{t \geq 1995\}\Big)\delta(k), \quad \cdots \textit{linear with two breaks}
\end{aligned}
$$

Note that $\{\delta_t(k)\}_t$ are slopes for the type-specific time trend in outcome level. To evaluate each of the restriction specifications, we computed the mean-squared forecasting error, using the last three time periods: year 1997-1999. Firstly, we used year 1988-1996 ($T_0 = 8$) as training data and year 1997 as test data. Then, we used year 1988-1997 ($T_0 = 9$) as training data and year 1998 as test data. Lastly, we used year 1988-1998 ($T_0 = 10$) as training data and year 1999 as test data.

Table 1: Cross validation result with $K = 2$

MSFE	4.51	5.81	4.84	7.50
specification	Cons	Linear	Linear	Linear
# of breaks	-	0	1	2

Table 1 contains the mean-squared forecasting error of the $K = 2$ type classification using each smoothness restriction. Based on the cross validation result, we used the constant slope as our main empirical specification in the type classification step.

1.2 Sensitivity to the number of types

As a sensitivity analysis with regard to the number of types K, we considered the type classification under $K = 3, 4$ in addition to $K = 2$. To assess the sensitivity of the classification result to the number of types K, we firstly report the Bayesian information criterion for each value of K as suggested in Bonhomme and Manresa (2015); Janys and Siflinger (2024):

$$\frac{1}{nT_0}\sum_{i,t}\left(Y_{it} - Y_{it-1} - \hat{\delta}_t(\hat{k}_i) - X_{it}^{\intercal}\hat{\theta}\right)^2 + \hat{\sigma}^2\frac{K+n+p}{nT_0}\log nT_0$$

where $\hat{\sigma}^2$ is estimated with the largest $K = 4$.[1] Table 2 contains the information criterion for each number of types $K = 2, 3, 4$. Secondly, we compare the classification results across the different number of types. Table 3 finds seven groups of units depending on how their type

[1]The constant slope restriction $\delta_t(k) = \delta(k)$ is imposed for $K = 3$ and $K = 4$ as well and thus the number of parameter is set to be $K + n + p$: K constant slopes $\{\delta(k)\}_k$, n types $\{k_i\}_{i=1}^n$ and p control covariate coefficients θ. If the type-specific time fixed-effects were allowed to be fully heterogeneous across t, the number of parameters would be $KT_0 + n + p$.

estimate changes along with $K = 2, 3, 4$. For comparison across K, we reorder the types in the decreasing order of $\delta(k)$; the dissimilarity index rose the fastest for Type 1.

Table 2: BIC across $K = 2, 3, 4$

K	2	3	4
BIC	12.237	12.154	12.160

Table 3: Type classification comparison betweeon $K = 2$ and $K = 3$

<table>
<tr><td>Type seq.</td><td>$(1,1,1)$</td><td>$(1,1,2)$</td><td>$(1,2,2)$</td><td>$(2,2,2)$</td><td>$(2,2,3)$</td><td>$(2,3,3)$</td><td>$(2,3,4)$</td></tr>
<tr><td>$K = 2$</td><td colspan="3">1</td><td colspan="4">2</td></tr>
<tr><td>$K = 3$</td><td colspan="2">1</td><td colspan="3">2</td><td colspan="2">3</td></tr>
<tr><td>$K = 4$</td><td>1</td><td colspan="3">2</td><td colspan="2">3</td><td>4</td></tr>
<tr><td># of units</td><td>9</td><td>10</td><td>3</td><td>9</td><td>12</td><td>3</td><td>4</td></tr>
</table>

Each column denotes a sequence of type estimates as K changes. For example, the first column finds number of units who were assigned to Type 1 in all of the three type classification results.

Table 2 suggests that the classification may suffer from overfitting when larger number of types is used: $K = 4$. In line with this observation, Table 3 shows us that increasing the number of types gives us types where only a few number of units are assigned: e.g., Type 4 when $K = 4$.

In the rest of the subsection, we report the descriptive statistics and the treatment effect estimates for $K = 3$ and $K = 4$. Table 4 and Table 5 contain within-type balancedness tests for $K = 3$ while Table 6 and Table 7 contain within-type balancedness tests for $K = 4$. Within each type, the control covariates are well-balanced across treatment status. Figure 1 and Figure 2 contain the treatment effect estimation results, respectively for $K = 3$ and $K = 4$. Similarly to $K = 2$ case, we find bigger treatment effect for school districts where the dissimilarity index in the pretreatment periods was rising faster. Lastly, Table 8 and Table 9 contain descriptive statistics for each type.

Table 4: Within-type Balancedness Test, $t = 1988$, $K = 3$

Type 1	treated	never-treated	Diff
$\mathbf{1}\{\text{central city}\}$	0.29	0.58	-0.30
	(0.49)	(0.51)	(0.24)
% (white)	58.20	61.50	-3.30
	(19.05)	(21.29)	(9.47)
% (hispanic)	6.69	4.10	2.59
	(13.73)	(7.38)	(5.61)
% (free/reduced-price lunch)	39.71	37.80	1.91
	(10.82)	(17.95)	(6.60)
# (student)	47574	61230	-13656
	(30851)	(111488)	(34231)
N	7	12	-
p-value			0.519

Type 2	treated	never-treated	Diff
$\mathbf{1}\{\text{central city}\}$	0.67	0.58	0.08
	(0.49)	(0.51)	(0.21)
% (white)	44.54	55.68	-11.13
	(21.83)	(20.05)	(8.56)
% (hispanic)	17.19	9.65	7.54
	(17.00)	(11.46)	(5.92)
% (free/reduced-price lunch)	37.87	34.80	3.07
	(15.83)	(17.67)	(6.85)
# (student)	84552	45499	39053
	(73316)	(50724)	(25736)
N	12	12	-
p-value			0.591

The table reports means of the school district characteristics and their differences across treatment status within each type. The *p*-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros.

Table 5: Within-type Balancedness Test, $t = 1988$, $K = 3$

Type 3	treated	never-treated	Diff
$\mathbf{1}\{\text{central city}\}$	0.50	0.80	-0.30
	(0.71)	(0.45)	(0.54)
% (white)	70.18	40.36	-2.98
	(3.68)	(25.51)	(11.70)
% (hispanic)	9.24	29.43	-20.19
	(7.02)	(26.92)	(13.02)
% (free/reduced-price lunch)	33.47	44.97	-11.51
	(0.68)	(17.54)	(7.86)
# (student)	39196	139532	-100336
	(34375)	(262980)	(120094)
N	2	5	-
p-value			0.930

The table reports means of the school district characteristics and their differences across treatment status within each type. The p-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros.

Table 6: Within-type Balancedness Test, $t = 1988$, $K = 4$

Type 1	treated	never-treated	Diff
$\mathbf{1}\{\text{central city}\}$	0.33	0.50	-0.17
	(0.58)	(0.55)	(0.40)
% (white)	49.63	52.51	-2.87
	(16.76)	(26.71)	(14.58)
% (hispanic)	12.58	6.51	6.07
	(21.40)	(10.12)	(13.03)
% (free/reduced-price lunch)	46.07	35.84	10.23
	(14.40)	(22.25)	(12.31)
# (student)	44764	101883	-57119
	(40819)	(152725)	(66655)
N	3	6	-
p-value			0.6841

Type 2	treated	never-treated	Diff
$\mathbf{1}\{\text{central city}\}$	0.50	0.50	0.00
	(0.53)	(0.52)	(0.22)
% (white)	44.89	66.79	-21.90
	(24.53)	(13.75)	(8.72)
% (hispanic)	16.56	8.46	8.11
	(17.22)	(11.74)	(6.42)
% (free/reduced-price lunch)	37.95	34.17	3.78
	(12.03)	(15.81)	(5.94)
# (student)	90673	31723	58949
	(77156)	(22721)	(25265)
N	10	12	-
p-value			0.103

The table reports means of the school district characteristics and their differences across treatment status within each type. The p-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros.

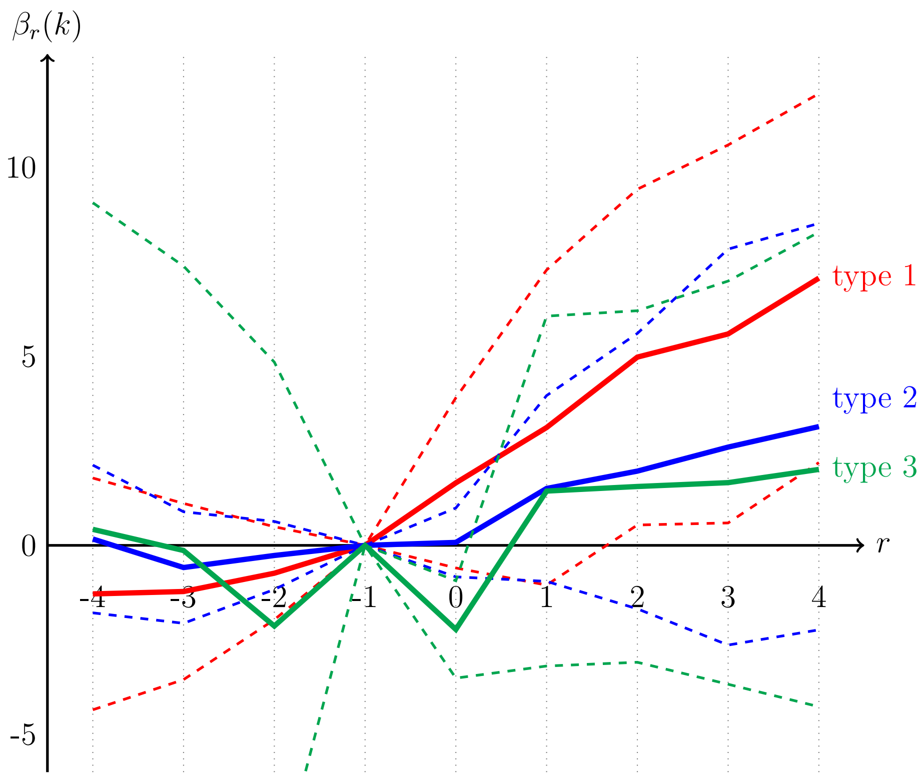
Table 7: Within-type Balancedness Test, $t = 1988$, $K = 4$

Type 3	treated	never-treated	Diff
$\mathbf{1}\{\text{central city}\}$	0.67	0.89	-0.22
	(0.52)	(0.33)	(0.24)
% (white)	57.35	41.20	16.16
	(18.34)	(22.73)	(10.65)
% (hispanic)	8.29	18.77	-10.48
	(13.81)	(23.11)	(9.55)
% (free/reduced-price lunch)	35.79	44.66	-8.87
	(17.54)	(17.09)	(9.15)
# (student)	51105	104913	-53808
	(33659)	(197483)	(67247)
N	6	9	-
p-value			0.863

Type 4	treated	never-treated	Diff
$\mathbf{1}\{\text{central city}\}$	0.50	0.50	0.00
	(0.71)	(0.71)	(0.71)
% (white)	70.18	60.28	9.90
	(3.68)	(18.37)	(13.25)
% (hispanic)	9.24	1.34	7.90
	(7.02)	(1.72)	(5.11)
% (free/reduced-price lunch)	33.47	34.51	-1.05
	(6.82)	(19.66)	(13.91)
# (student)	39196	21109	18087
	(34375)	(15180)	(26572)
N	2	2	-

The table reports means of the school district characteristics and their differences across treatment status within each type. The p-value is for the null hypothesis that the means of differences between treated units and never-treated units are all zeros; there are too few units in Type 4 for within-type balancedness test so there is no p-value reported for Type 4.

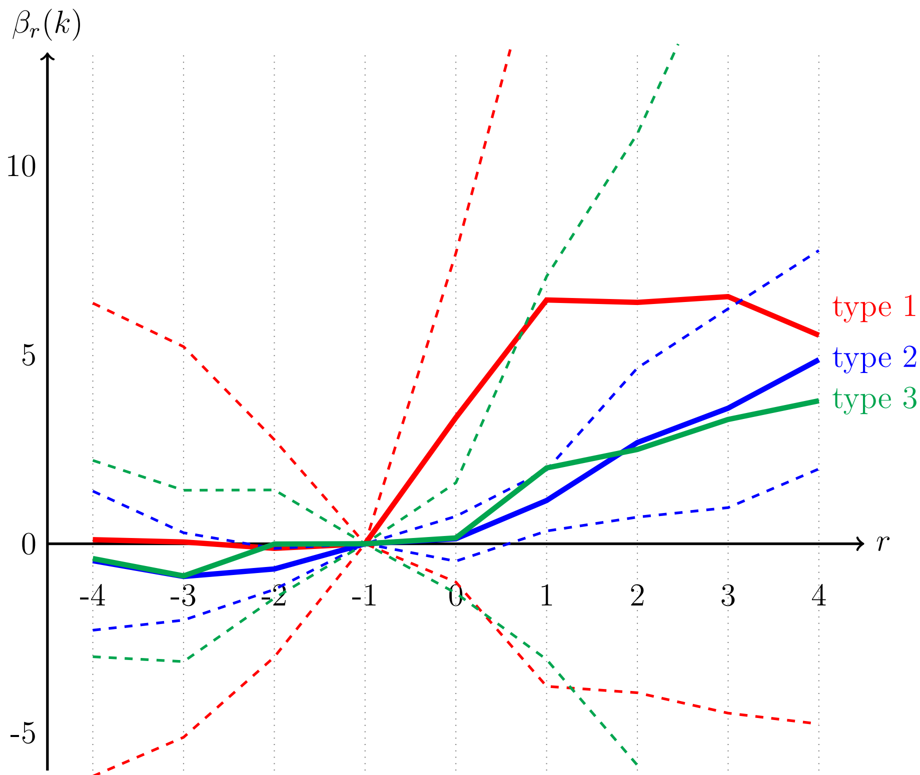
Figure 1: Type-specific CATT, $K = 3$



The graph reports the type-specific diff-in-diff estimates for the effect of dismissing court-mandated desegregation plan on the dissimilarity index of a school district. The dissimilarity index ranges from 0 to 100. In 1988, the average dissimilarity index was 34 and the standard deviation was 16.

Types are ordered in the decreasing order of $\delta(k)$; the dissimilarity index rose the fastest for Type 1 and the slowest for Type 3. The dashed lines denote the confidence intervals at 0.05 significance level, computed with asymptotic standard errors.

Figure 2: Type-specific CATT, $K = 4$



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Types are ordered in the decreasing order of $\delta(k)$; the dissimilarity index rose the fastest for Type 1 and the slowest for Type 3. The dashed lines denote the confidence intervals at 0.05 significance level, computed with asymptotic standard errors. The treatment effect estimates for Type 4 are omitted since there are too few units in Type 4 and therefore the esitmates are not as precisely estimated as for other types.

Table 8: Type-specific Descriptive Statistics, $t = 1988$, $K = 3$

	Type 1	Type 2	Type 3
dissimilarity index	28.21	37.41	39.79
	(13.32)	(18.51)	(16.04)
$\mathbf{1}\{\text{central city}\}$	0.47	0.63	0.71
	(0.51)	(0.49)	(0.49)
% (white)	60.28	50.11	48.88
	(20.02)	(21.27)	(25.45)
% (hispanic)	5.05	13.42	23.66
	(9.89)	(14.69)	(24.25)
% (free/reduced-price lunch)	38.51	36.34	41.68
	(15.39)	(16.48)	(15.38)
# (student)	56199	65026	110865
	(89213)	(64801)	(220680)
N	19	24	7

The table reports the group means of the school district characteristics and their differences. The p-value for the null hypothesis that Type 1 and Type 2 share the same mean is 0.001. The p-values for the null hypothesis that Type 1 and Type 3 share the same mean is 0.218 and that for Type 2 and Type 3 is 0.804.

Table 9: Type-specific Descriptive Statistics, $t = 1988$, $K = 4$

	Type 1	Type 2	Type 3	Type 4
dissimilarity index	27.11	33.60	39.43	34.45
	(16.50)	(13.75)	(21.14)	(12.83)
$\mathbf{1}\{\text{central city}\}$	0.44	0.50	0.80	0.50
	(0.52)	(0.51)	(0.41)	(0.58)
% (white)	51.55	56.83	47.66	65.23
	(22.76)	(21.95)	(21.97)	(12.24)
% (hispanic)	8.53	12.14	14.58	5.29
	(13.70)	(14.71)	(20.04)	(6.18)
% (free/reduced-price lunch)	39.25	35.89	41.12	33.99
	(19.68)	(14.02)	(17.23)	(11.37)
# (student)	86844	58518	83389	30153
	(125739)	(61027)	(153084)	(24078)
N	9	22	15	4

The table reports the group means of the school district characteristics and their differences. The null hypothesis that two types share the same mean is not rejected at the 0.05 significance level for any pair of two types, possibly due to small number of units per type.

1.3 Type classification

In this subsection, we present the full classification result for $K = 2$. Below are the numbers of school districts in each states for Type 1 and Type 2. The number of treated school districts are denoted with red while the the number of never-treated school districts are denoted with black. Table 10 further summarizes the list and presents the number of school districts for each census region. The classification result suggests that the type classification captures heterogeneity across units that is not fully explained by the geographical location; it appears that the location is not a strong predictor of the school district's type.

Type 1 Alabama (3), Arkansas (1), Florida (2/4), Illinois (1), Kentucky (1/1), Mississippi (1), New York (1), North Carolina (3), Ohio (1), Pennsylvania (1), Texas (1), Wisconsin (1)

Type 2 Alabama (1), Arizona (1), Arkansas (1), California (2/1), Connecticut (2), Florida (5), Indiana (1), Maryland (1), Michigan (2/1), Mississippi (2), North Carolina (1), Pennsylvania (1), Texas (3/3)

	Northeast	Midwest	South	West
Type 1	2	2/1	10/7	-
Type 2	3	3/1	7/10	2/2

Table 10: Distribution of types across census regions

As a robustness check on the classification, we additionally conducted a type classification exercise only on the never-treated units. Out of the 50 school district, 29 school districts were never dismissed of the court-mandated desegregation plan until 2007, effectively giving us 19 untreated outcomes. The type classification was firstly done with the 29 never-treated units only, using $T = 19$, and then extrapolated to the 21 treated units, using all the available pretreatment outcomes for each unit. Since the number of time periods we use is longer, we considered one additional smoothness restriction:

$$\delta_t(k) = \delta(k), \quad \cdots \textit{constant}$$
$$\delta_t(k) = \Big(1, t-1989\Big)\delta(k), \quad \cdots \textit{linear}$$
$$\delta_t(k) = \Big(1, t-1989, (t-1998)\mathbf{1}\{t \geq 1998\}\Big)\delta(k), \quad \cdots \textit{linear with a break}$$
$$\delta_t(k) = \Big(1, t-1989, (t-1995)\mathbf{1}\{t \geq 1995\}, (t-2001)\mathbf{1}\{t \geq 2001\}\Big)\delta(k), \quad \cdots \textit{linear with two breaks}$$
$$\delta_t(k) = \Big(1, t-1989, (t-1989)^2\Big)\delta(k), \quad \cdots \textit{quadratic}$$

Table 11 conatins the cross validation results; the linear specification without a break is selected based on the mean squared forecasting error using the last three periods (year 2005-2007).

MSFE	2.674	2.646	2.650	2.792	3.220
specification	Cons	Linear	Linear	Linear	Quad
# of breaks	-	0	1	2	0

Table 11: Cross validation result with $K = 2$, never-treated units only

Table 12 compares the two classification results, suggesting that the pretreatment time periods ($T_0 = 11$) contain sufficient information for classification.

	1	2	total
1	22	0	22
2	1	27	28
total	23	27	50

Table 12: Counts of school districts for each type

The rows are the types estimated with the population pretreatment outcomes and the columns are the types estimated with the never-treated units and extrapolated to the treated units.

2 Proof for Theorem 2

In the proof sections, we will use the dot notation to denote the first difference: $\dot{Y}_{it} = Y_{it} - Y_{it-1}$, $\dot{X}_{it} = X_{it} - X_{it-1}$ and $\dot{U}_{it} = U_{it} - U_{it-1}$. Also, we will use the superscript naught to denote the true values of the parameters and the latent type variable: e.g. k_i^0 is the true type of unit i.

We prove Theorem 2 in the context of a linear model for outcome in level (see *Remark 5* of the main text). This subsumes the case of a linear model for first-differenced outcomes, by replacing $\dot{X}_{it}$ and $\dot{U}_{it}$ with X_{it} and U_{it}. For Theorem 1, replace $\delta_t^0(k)$ and $\dot{U}_{it}$ with $\mathbf{E}[\dot{Y}_{it}(\infty)|k]$ and $\dot{Y}_{it}(E_i) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0]$.

Step 1

The first step is to obtain an approximation of the objective function. Note that

$$\begin{aligned}
\widehat{Q}(\theta, \delta, \gamma) &= \frac{1}{nT_0} \sum_{i=1}^{n} \sum_{t=-T_0}^{-1} \left(\dot{Y}_{it} - \delta_t(k_i) - \dot{X}_{it}^\intercal \theta \right)^2 \\
&= \frac{1}{nT_0} \sum_{i,t} \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^\intercal (\theta^0 - \theta) + \dot{U}_{it} \right)^2 \\
&= \frac{1}{nT_0} \sum_{i,t} \left\{ \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^\intercal (\theta^0 - \theta) \right)^2 + \dot{U}_{it}^2 \right\} \\
&\quad + \frac{2}{nT_0} \sum_{i,t} \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^\intercal (\theta^0 - \theta) \right) \dot{U}_{it}.
\end{aligned}$$

Let

$$\tilde{Q}(\theta, \delta, \gamma) = \frac{1}{nT_0} \sum_{i,t} \left\{ \left(\delta_t^0(k_i^0) - \delta_t(k_i) + \dot{X}_{it}^\intercal (\theta^0 - \theta) \right)^2 + \dot{U}_{it}^2 \right\}.$$

Then,

$$\left|\widehat{Q}(\theta,\delta,\gamma)-\tilde{Q}(\theta,\delta,\gamma)\right| = \left|\frac{2}{nT_0}\sum_{i,t}\Big(\delta_t^0(k_i^0)-\delta_t(k_i)+\dot{X}_{it}^{\intercal}(\theta^0-\theta)\Big)\dot{U}_{it}\right|$$
$$\leq \left|\frac{2}{nT_0}\sum_{i,t}\Big(\delta_t^0(k_i^0)-\delta_t(k_i)\Big)\dot{U}_{it}\right| + \left|\frac{2}{nT_0}\sum_{i,t}\dot{X}_{it}^{\intercal}(\theta^0-\theta)\dot{U}_{it}\right|. \quad (1)$$

Firstly, find that

$$\left|\frac{1}{nT_0}\sum_{i,t}\delta_t^0(k_i^0)\dot{U}_{it}\right| \leq \sum_{k=1}^{K}\left|\frac{1}{nT_0}\sum_{i,t}\delta_t^0(k)\dot{U}_{it}\mathbf{1}\{k_i^0=k\}\right|$$
$$\leq \sum_{k=1}^{K}\left(\frac{1}{T_0}\sum_t \delta_t^0(k)^2\right)^{\frac{1}{2}}\left(\frac{1}{T_0}\sum_t\left(\frac{1}{n}\sum_i \dot{U}_{it}\mathbf{1}\{k_i^0=k\}\right)^2\right)^{\frac{1}{2}}$$
$$\leq M\sum_{k=1}^{K}\left(\frac{1}{n^2T_0}\sum_{i,j,t}\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}\right)^{\frac{1}{2}} \xrightarrow{p} 0.$$

The first two inequalities are from separating the summation into types and applying Cauchy-Schwartz's inequality to over t. The third is from Assumption 7-b. It remains to prove the convergence in probability; for that we use Assumption 7-a,d. With some constant $C>0$ that only depends on $M>0$ from Assumption 7,

$$\mathbf{E}\left[\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}\right] = \begin{cases}\mathbf{E}\big[\dot{U}_{it}^2\mathbf{1}\{k_i^0=k\}\big]\leq C & \text{if } i=j\\ \mathbf{E}\left[\dot{U}_{it}\mathbf{1}\{k_i^0=k\}\right]\mathbf{E}\left[\dot{U}_{jt}\mathbf{1}\{k_j^0=k\}\right]=0 & \text{if } i\neq j\end{cases}$$

since $\mathbf{E}[\dot{U}_{it}\mathbf{1}\{k_i^0=k\}]=\mathbf{E}[\dot{U}_{it}|k_i^0=k]\Pr\{k_i^0=k\}=0$.[2] Then,

$$\mathbf{E}\left[\frac{1}{nT_0}\sum_{i,j,t}\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}\right]\leq C.$$

[2]In the case of Theorem 1,

$$\mathbf{E}[\dot{Y}_{it}(E_i)-\mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0]|k_i^0=k]=\mathbf{E}[\mathbf{E}[\dot{Y}_{it}(E_i)-\dot{Y}_{it}(\infty)|k_i^0,E_i]|k_i^0=k]=0$$

from Assumption 2.

and $\frac{1}{n^2T_0}\sum_{i,j,t}\dot{U}_{it}\dot{U}_{jt}\mathbf{1}\{k_i^0=k_j^0=k\}=o_p(1)$. We can repeat this for the other quantity in the first term of (1).

Secondly, again from applying Cauchy-Schwartz's inequality and Jensen's inequality,

$$\begin{aligned}\left|\frac{1}{nT_0}\sum_{i,t}\dot{X}_{it}^{\intercal}\left(\theta^0-\theta\right)\dot{U}_{it}\right| &\leq \frac{1}{T_0}\sum_{t}\left\|\frac{1}{n}\sum_{i}\dot{U}_{it}\dot{X}_{it}\right\|_2\cdot\left\|\theta^0-\theta\right\|_2 \\ &\leq \frac{2M}{\sqrt{n}}\cdot\frac{1}{T_0}\sum_{t}\left(\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right)^{\frac{1}{2}}=\frac{2M}{\sqrt{n}}\cdot O_p(1)\xrightarrow{p}0\end{aligned}$$

The convergence in probability is from Assumption 7-a,d. Find that

$$\mathbf{E}\left[\dot{U}_{it}\dot{X}_{it}\right]=\mathbf{0},\qquad\qquad \mathbf{E}\left[\dot{U}_{it}^2\dot{X}_{it}^{\intercal}\dot{X}_{it}\right]\leq C$$

with some constant $C>0$ that only depends on $M>0$ from Assumption 7. Thus,

$$\frac{1}{T_0}\sum_{t}\mathbf{E}\left[\left(\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right)^{\frac{1}{2}}\right]\leq\frac{1}{T_0}\sum_{t}\left(\mathbf{E}\left[\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right]\right)^{\frac{1}{2}}\leq\sqrt{C}.$$

Then $\frac{1}{T_0}\sum_t\left(\frac{1}{n}\sum_{i,j}\dot{U}_{it}\dot{U}_{jt}\dot{X}_{it}^{\intercal}\dot{X}_{jt}\right)^{\frac{1}{2}}=O_p(1)$ and $\widehat{Q}(\theta,\delta,\gamma)-\tilde{Q}(\theta,\delta,\gamma)=o_p(1)$.

Step 2

By plugging in the true parameters, $\tilde{Q}(\theta^0,\delta^0,\gamma^0)=\frac{1}{nT_0}\sum_{i,t}\dot{U}_{it}^2$ and

$$\begin{aligned}\tilde{Q}(\theta,\delta,\gamma)-\tilde{Q}(\theta^0,\delta^0,\gamma^0) &= \frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0)-\delta_t(k_i)+\dot{X}_{it}^{\intercal}(\theta^0-\theta)\right)^2 \\ &\geq \frac{1}{nT_0}\sum_{i,t}\left(\dot{X}_{it}^{\intercal}(\theta^0-\theta)-\bar{\dot{X}}_{k_i^0\wedge k_i,t}{}^{\intercal}(\theta^0-\theta)\right)^2 \\ &= \frac{1}{nT_0}\sum_{i,t}(\theta^0-\theta)^{\intercal}\left(\dot{X}_{it}-\bar{\dot{X}}_{k_i^0\wedge k_i,t}\right)\left(\dot{X}_{it}-\bar{\dot{X}}_{k_i^0\wedge k_i,t}\right)^{\intercal}(\theta^0-\theta) \\ &\geq \min_{\gamma\in\Gamma}\rho_n(\gamma)\cdot\|\theta^0-\theta\|_2^2.\end{aligned}$$

Note that the unknowns in $\tilde{Q}(\theta, \delta, \gamma) - \tilde{Q}(\theta^0, \delta^0, \gamma^0)$ other than $(\theta^0 - \theta)$ are functions of (t, k_i^0, k_i). Thus, subtracting the group mean defined with (t, k_i^0, k_i) from $\dot{X}_{it}^\intercal(\theta^0 - \theta)$ is the lower bound for the sum of squares, giving us the first inequality.

Since the estimator minimizes the objective function,

$$\begin{aligned}
\tilde{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) &= \widehat{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) + o_p(1) \\
&\leq \widehat{Q}(\theta^0, \delta^0, \gamma^0) + o_p(1) \\
&= \tilde{Q}(\theta^0, \delta^0, \gamma^0) + o_p(1).
\end{aligned}$$

Therefore from Assumption 7-h,

$$\begin{aligned}
\min_{\gamma\in\Gamma} \rho_n(\gamma) \cdot \left\|\theta^0 - \hat{\theta}\right\|_2^2 &\leq \tilde{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) - \tilde{Q}(\theta^0, \delta^0, \gamma^0) = o_p(1) \\
\left\|\theta^0 - \hat{\theta}\right\|_2^2 &= \frac{1}{\min_{\gamma\in\Gamma} \rho_n(\gamma)} \cdot \min_{\gamma\in\Gamma} \rho_n(\gamma) \left\|\theta^0 - \hat{\theta}\right\|_2^2 \xrightarrow{p} \frac{1}{\rho} \cdot 0 = 0.
\end{aligned}$$

We have consistency of $\hat{\theta}$.

Step 3

In this step, we show that $\{\hat{\delta}_t(\hat{k}_i)\}_{i,t}$ is close to $\{\delta_t^0(k_i^0)\}_{i,t}$ in terms of the l_2 norm.

$$\begin{aligned}
&\left|\tilde{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) - \tilde{Q}(\theta^0, \hat{\delta}, \hat{\gamma})\right| \\
&= \left|\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i) + \dot{X}_{it}^\intercal(\theta^0 - \hat{\theta})\right)^2 - \frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)^2\right| \\
&\leq \left|\frac{2}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)\dot{X}_{it}^\intercal(\theta^0 - \hat{\theta}) + \frac{1}{nT_0}\sum_{i,t}\left(\dot{X}_{it}^\intercal(\theta^0 - \hat{\theta})\right)^2\right| \\
&\leq \frac{4M}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2 \cdot \left\|\theta^0 - \hat{\theta}\right\|_2 + \frac{1}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2^2 \cdot \left\|\theta^0 - \hat{\theta}\right\|_2^2 = o_p(1).
\end{aligned}$$

The second inequality is from Assumption 7-b and Cauchy-Schwartz inequality on $|\dot{X}_{it}^{\intercal}(\theta^0 - \hat{\theta})|$. Note that for any n, $\frac{1}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2^2$ is bounded in expectation by $4M$ from Assumption 7.d and thus $O_p(1)$. Likewise, $\frac{1}{nT_0}\sum_{i,t}\|\dot{X}_{it}\|_2$ is bounded in expectation by $2\sqrt{M}$. Since we have shown $\hat{\theta} \xrightarrow{p} \theta^0$, we have the last equality. Then,

$$\begin{aligned}
&\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)^2 + \frac{1}{nT_0}\sum_{i,t}\dot{U}_{it}^2 \\
&= \tilde{Q}(\theta^0, \hat{\delta}, \hat{\gamma}) = \tilde{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) + o_p(1) = \widehat{Q}(\hat{\theta}, \hat{\delta}, \hat{\gamma}) + o_p(1) \\
&\leq \widehat{Q}(\theta^0, \delta^0, \gamma^0) + o_p(1) = \frac{1}{nT_0}\sum_{i,t}\dot{U}_{it}^2 + o_p(1).
\end{aligned}$$

$\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)^2 = o_p(1)$. For Theorem 1, the result holds directly from Step 1.

Step 4

In this step, we find some permutation on $\left\{\hat{\delta}_t(k)\right\}_{t,k}$ so that $\frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(k_i^0)\right)^2$ is close to zero. Note that $\widehat{Q}(\theta, \delta, \gamma)$ does not vary for any $(\theta, \tilde{\delta}, \tilde{\gamma})$ defined with a permutation on $(1, \cdots, K)$: with σ, a permutation on $\{1, \cdots, K\}$, letting $\tilde{k}_i = \sigma(k_i)$ and $\tilde{\delta}_t(\sigma(k)) = \delta_t(k)$ gives us $\widehat{Q}(\theta, \delta, \gamma) = \widehat{Q}(\theta, \tilde{\delta}, \tilde{\gamma})$. Thus, we want to define a bijection on $\{1, \cdots, K\}$ to match $\hat{k}$ with true k^0, to have classification result. Define a function σ by letting

$$\sigma(k) = \arg\min_{\tilde{k}} \frac{1}{T_0}\sum_{t=-T_0}^{-1}\left(\delta_t^0(k) - \hat{\delta}_t(\sigma(k))\right)^2$$

for each k. First, let us show that σ actually lets the objective go to zero for each k: fix k,

$$\begin{aligned}
&\min_{\tilde{k}} \frac{1}{T_0}\sum_{t=-T_0}^{-1}\left(\delta_t^0(k) - \hat{\delta}_t(\sigma(k))\right)^2 \\
&\leq \frac{n}{\sum_i \mathbf{1}\{k_i^0 = k\}} \cdot \min_{\tilde{k}} \frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k) - \hat{\delta}_t(\sigma(k))\right)^2 \mathbf{1}\{k_i^0 = k\} \\
&\leq \frac{n}{\sum_i \mathbf{1}\{k_i^0 = k\}} \cdot \frac{1}{nT_0}\sum_{i,t}\left(\delta_t^0(k_i^0) - \hat{\delta}_t(\hat{k}_i)\right)^2 \xrightarrow{p} 0
\end{aligned}$$

as $n \to \infty$. From Assumption 7-f, we have the convergence.

For some $k, \tilde{k}$ such that $k \neq \tilde{k}$,

$$
\begin{aligned}
&\left(\frac{1}{T_0}\sum_t \left(\hat{\delta}_t(\sigma(k)) - \hat{\delta}_t(\sigma(\tilde{k}))\right)^2\right)^{\frac{1}{2}} \\
&\geq \left(\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2\right)^{\frac{1}{2}} \\
&\qquad - \left(\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(\sigma(k))\right)^2\right)^{\frac{1}{2}} - \left(\frac{1}{T_0}\sum_t \left(\delta_t^0(\tilde{k}) - \hat{\delta}_t(\sigma(\tilde{k}))\right)^2\right)^{\frac{1}{2}} \\
&\xrightarrow{p} c(k, \tilde{k}) > 0
\end{aligned}
$$

from Assumption 7.c. Thus, $\Pr\{\sigma \text{ is not bijective}\} \leq \sum_{k \neq \tilde{k}} \Pr\left\{\sigma(k) = \sigma(\tilde{k})\right\} \to 0$ as $n \to \infty$. Note that σ depends on the dataset.

Before proceeding to the next step, let us drop the σ notation. Based on σ, we can construct a bijection $\tilde{\sigma} : \{1, \cdots, K\} \to \{1, \cdots, K\}$ such that

$$
\frac{1}{T}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(\tilde{\sigma}(k))\right)^2 \xrightarrow{p} 0 \tag{2}
$$

as $n \to \infty$ for all k, by letting $\tilde{\sigma} = \sigma$ whenever σ is bijective. From now on, I will drop $\tilde{\sigma}$ by always rearranging $(\hat{\theta}, \hat{\delta}, \hat{\gamma})$ so that $\tilde{\sigma}(k) = k$.

Step 5

Here, we study the probability of the K-means algorithm assigning a wrong type to an arbitrary unit i.

$$
\begin{aligned}
\Pr\left\{\hat{k}_i \neq k_i^0\right\} &\leq \sum_{\tilde{k} \neq k_i^0} \Pr\left\{\frac{1}{T_0}\sum_t \left(\dot{Y}_{it} - \hat{\delta}_t(\tilde{k}) - \dot{X}_{it}^\intercal \hat{\theta}\right)^2 \leq \frac{1}{T_0}\sum_t \left(\dot{Y}_{it} - \hat{\delta}_t(k_i^0) - \dot{X}_{it}^\intercal \hat{\theta}\right)^2\right\} \\
&= \sum_{\tilde{k} \neq k_i^0} \Pr\left\{\frac{2}{T_0}\sum_t \left(\hat{\delta}_t(k_i^0) - \hat{\delta}_t(\tilde{k})\right) \cdot \left(\dot{Y}_{it} - \frac{\hat{\delta}_t(k_i^0) + \hat{\delta}_t(\tilde{k})}{2} - \dot{X}_{it}^\intercal \hat{\theta}\right) \leq 0\right\}.
\end{aligned}
$$

The inequality is from the second stage of the K-means algorithm. Then,

$$
\begin{aligned}
&\Pr\left\{\hat{k}_i \neq k_i^0\right\} \\
&= \sum_{\tilde{k}\neq k_i^0} \Pr\left\{\frac{2}{T_0}\sum_t \left(\hat{\delta}_t(k_i^0) - \hat{\delta}_t(\tilde{k})\right)\cdot\left(\delta_t^0(k_i^0) - \frac{\hat{\delta}_t(k_i^0)+\hat{\delta}_t(\tilde{k})}{2} + \dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta}) + \dot{U}_{it}\right) \leq 0\right\} \\
&\leq \sum_k \sum_{\tilde{k}\neq k} \Pr\left\{\frac{2}{T}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\cdot\left(\delta_t^0(k) - \frac{\hat{\delta}_t(k)+\hat{\delta}_t(\tilde{k})}{2} + \dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta}) + \dot{U}_{it}\right) \leq 0\right\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
A_{ik\tilde{k}} &= \frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\dot{U}_{it} + \frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta}) \\
&\quad + \frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\cdot\left(\dot{\delta}_t^0(k) - \frac{\hat{\delta}_t(k)+\hat{\delta}_t(\tilde{k})}{2}\right) \\
B_{ik\tilde{k}} &= \frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} + \frac{1}{2T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2.
\end{aligned}
$$

Note that $A_{ik\tilde{k}}$ depends on the estimator $(\hat{\theta}, \hat{\delta}, \hat{\gamma})$ while $B_{ik\tilde{k}}$ does not. Then,

$$
\Pr\left\{\hat{k}_i \neq k_i^0\right\} \leq \sum_k \sum_{\tilde{k}\neq k} \Pr\left\{A_{ik\tilde{k}} \leq 0\right\} \leq \sum_k \sum_{\tilde{k}\neq k} \Pr\left\{B_{ik\tilde{k}} \leq |B_{ik\tilde{k}} - A_{ik\tilde{k}}|\right\} \tag{3}
$$

We will show that $A_{ik\tilde{k}}$ and $B_{ik\tilde{k}}$ are sufficiently close to each other and that $\Pr\left\{B_{ik\tilde{k}} \leq 0\right\}$ converges to zero sufficiently fast.

$$
\begin{aligned}
|B_{ik\tilde{k}} - A_{ik\tilde{k}}| &\leq \left|\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(k)\right)\dot{U}_{it}\right| + \left|\frac{1}{T_0}\sum_t \left(\delta_t^0(\tilde{k}) - \hat{\delta}_t(\tilde{k})\right)\dot{U}_{it}\right| \\
&\quad + \left|\frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta})\right| \\
&\quad + \left|\frac{1}{2T_0}\sum_t \left(\delta_t^0(k) - \hat{\delta}_t(k)\right)\cdot\left(-\delta_t^0(k) + \delta_t^0(\tilde{k}) + \hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\right| \\
&\quad + \left|\frac{1}{2T_0}\sum_t \left(\delta_t^0(\tilde{k}) - \hat{\delta}_t(\tilde{k})\right)\cdot\left(\delta_t^0(k) - \delta_t^0(\tilde{k}) + +\hat{\delta}_t(k) - \hat{\delta}_t(\tilde{k})\right)\right|.
\end{aligned}
$$

We apply Cauchy-Schwartz's inequality to each of the five terms so that we can use the consistency result in (2). For the first term,

$$\left|\frac{1}{T_0}\sum_t \left(\delta_t^0(k)-\hat{\delta}_t(k)\right)\dot{U}_{it}\right| \le \left(\frac{1}{T_0}\sum_t \left(\delta_t^0(k)-\hat{\delta}_t(k)\right)^2\right)^{\frac{1}{2}} \left(\frac{1}{T_0}\sum_t \dot{U}_{it}^2\right)^{\frac{1}{2}}$$

and similarly for the second term. As for the third term, from Assumption 7-b,

$$\begin{aligned}\left|\frac{1}{T_0}\sum_t \left(\hat{\delta}_t(k)-\hat{\delta}_t(\tilde{k})\right)\dot{X}_{it}^{\intercal}(\theta^0-\hat{\theta})\right| &\le \frac{1}{T_0}\sum_t \left|\hat{\delta}_t(k)-\hat{\delta}_t(\tilde{k})\right|\cdot\|\dot{X}_{it}\|_2\cdot\left\|\theta^0-\hat{\theta}\right\|_2 \\ &\le 2M\left(\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2\right)\cdot\left\|\theta^0-\hat{\theta}\right\|_2\end{aligned}$$

Last, for the fourth term, from Assumption 7-b,

$$\begin{aligned}&\left|\frac{1}{2T_0}\sum_t \left(\delta_t^0(k)-\hat{\delta}_t(k)\right)\cdot\left(-\delta_t^0(k)+\delta_t^0(\tilde{k})+\hat{\delta}_t(k)-\hat{\delta}_t(\tilde{k})\right)\right| \\ &\le M\left(\frac{1}{T_0}\sum_t \left(\delta_t^0(k)-\hat{\delta}_t(k)\right)^2\right)^{\frac{1}{2}}\end{aligned}$$

and similarly for the fifth term. From Assumption 7-d, both $\frac{1}{T_0}\sum_t \dot{U}_{it}^2$ and $\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2$ are bounded in expectation by the same bound for every n and thus $O_p(1)$. To use (2), choose an arbitrary $\eta > 0$ and focus only on the event of

$$\left\|\theta^0-\hat{\theta}\right\|_2, \left(\frac{1}{T_0}\sum_t \left(\delta_t^0(k)-\hat{\delta}_t(k)\right)\right)^{\frac{1}{2}} < \eta \tag{4}$$

for all k. When (4) is true, with some constant $C > 0$,

$$|B_{ik\tilde{k}}-A_{ik\tilde{k}}| \le \eta C\left(\left(\frac{1}{T_0}\sum_t \dot{U}_{it}^2\right)^{\frac{1}{2}} + \frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 + 1\right).$$

Note that C only depend on M from Assumption 7 and does not depend on η. Let $D(\eta)$ be a binary random variable which equals one if (4) holds true for all k. Then,

$$\begin{aligned}
&\Pr\left\{B_{ik\tilde{k}} \leq \left|B_{ik\tilde{k}} - A_{ik\tilde{k}}\right|, D(\eta) = 1\right\} \\
&\leq \Pr\left\{B_{ik\tilde{k}} \leq \eta C\left(\left(\frac{1}{T_0}\sum_t \dot{U}_{it}^2\right)^{\frac{1}{2}} + \frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 + 1\right)\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq M^*\right\} + \Pr\left\{\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 \geq M^*\right\} \\
&\qquad + \Pr\left\{B_{ik\tilde{k}} \leq \eta C(M^* + \sqrt{M^*} + 1)\right\} \qquad (5)
\end{aligned}$$

for any arbitrary $M^* > 0$. Let $M^* = \max\{4\sqrt{M}+1, 4\tilde{M}\}$ since $\mathbf{E}[\dot{U}_{it}^2]$ is uniformly bounded by $4\sqrt{M}$ from Assumption 7-d.[3]

Now, we show that all three probabilities in (5) go to zero. For that, we use Lemma B5 of Bonhomme and Manresa (2015). For the first quantity, find that

$$\begin{aligned}
\Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq M^*\right\} &\leq \Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq 4\sqrt{M} + 1\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \left(\dot{U}_{it}^2 - \mathbf{E}\left[\dot{U}_{it}^2\right]\right) \geq 1\right\}.
\end{aligned}$$

Let $Z_t = \dot{U}_{it}^2 - \mathbf{E}[\dot{U}_{it}^2]$. WTS $\{Z_t\}_{t=1}^{T_0}$ satisfies the condition given in Assumption 7-g.

$$\begin{aligned}
\Pr\{|Z_t| \geq z\} &= \Pr\left\{|U_{it} - U_{it-1}| \geq \sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}\right\} + \Pr\left\{|U_{it} - U_{it-1}| \leq \sqrt{\mathbf{E}[\dot{U}_{it}^2] - z}\right\} \\
&\leq \Pr\left\{|U_{it}| \geq \frac{\sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}}{2}\right\} + \Pr\left\{|U_{it-1}| \geq \frac{\sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}}{2}\right\} + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\} \\
&\leq 2\exp\left(1 - \left(\frac{\sqrt{\mathbf{E}[\dot{U}_{it}^2] + z}}{2b}\right)^{d_2}\right) + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\} \\
&\leq 2\exp\left(1 - \left(\frac{z}{2b}\right)^{d_2}\right) + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\}.
\end{aligned}$$

[3]In cases of the linear model for first-differenced outcomes and Theorem 1, a similar uniform bound on $\mathbf{E}[U_{it}^2]$ and $\mathbf{E}[(\dot{Y}_{it}(E_i) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0])^2]$ can be found.

We want to find some $\tilde{b}$ and $\tilde{d}_2$ such that

$$\Pr\left\{|Z_t| \geq z\right\} \leq \exp\left(1 - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2}\right).$$

Note that the RHS crosses one when $z = \tilde{b}$. It suffices to show

$$2\exp\left(1 - \left(\frac{z}{2b}\right)^{d_2}\right) + \mathbf{1}\{z \leq \mathbf{E}[\dot{U}_{it}^2]\} \leq \exp\left(1 - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2}\right) \tag{6}$$

for $z \geq \tilde{b}$. Fix some $\tilde{d}_2 \in (0, d_2)$ and let

$$\tilde{b} = \max\left\{4\sqrt{M} + 1, 2b\left(1 + \log 2\right)^{\frac{1}{d_2}}, 2b\left(\frac{\tilde{d}_2}{d_2}\right)^{\frac{1}{\tilde{d}_2}}\right\}.$$

Since $\tilde{b} > \sqrt{M} \geq \mathbf{E}[\dot{U}_{it}^2]$, (6) for $z \geq \tilde{b}$ is equivalent with

$$\exp\left(\left(\frac{z}{2b}\right)^{d_2} - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2}\right) \geq 2 \qquad \Leftrightarrow \qquad \left(\frac{z}{2b}\right)^{d_2} - \left(\frac{z}{\tilde{b}}\right)^{\tilde{d}_2} \geq \log 2.$$

The inequality holds at $z = \tilde{b}$ and the LHS in the last inequality strictly increases in z since

$$\frac{d_2 z^{d_2 - 1}}{(2b)^{d_2}} - \frac{\tilde{d}_2 z^{\tilde{d}_2 - 1}}{\tilde{b}^{\tilde{d}_2}} = z^{\tilde{d}_2 - 1}\left(\frac{d_2}{(2b)^{d_2}} z^{d_2 - \tilde{d}_2} - \frac{\tilde{d}_2}{\tilde{b}^{\tilde{d}_2}}\right) \geq 0$$

for all $z \geq \tilde{b}$. Z_t is strongly mixing since $\dot{U}_{it}^2$ is a measurable function of (U_{it}, U_{it-1}). By adjusting a and d_1, we can satisfy Assumption 7-g for Z_t. Thus, from Lemma B5 of Bonhomme and Manresa (2015), for any $\nu > 0$,

$$T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t \dot{U}_{it}^2 \geq M^*\right\} = o(1).$$

For Theorem 1, find that a similar result holds with $\Pr\left\{\frac{1}{T_0}\sum_t(\dot{Y}_{it}(e) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0])^2 \geq M^*\right\}$. Since E_i has finite support, $T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t(\dot{Y}_{it}(E_i) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0])^2 \geq M^*\right\} = o(1)$.

For the second quantity, find that

$$\Pr\left\{\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 \geq M^*\right\} \leq \Pr\left\{\frac{2}{T_0}\sum_{t=-T_0-1}^{-1}\|X_{it}\|_2 \geq 4\tilde{M}\right\}$$
$$\leq \Pr\left\{\frac{1}{T_0+1}\sum_{t=-T_0-1}^{-1}\|X_{it}\|_2 \geq \tilde{M}\right\}$$

From Assumption 7-d, for any $\nu > 0$,

$$T_0{}^{\nu}\Pr\left\{\frac{1}{T_0}\sum_t \|\dot{X}_{it}\|_2 \geq M^*\right\} = o(1).$$

For the last quantity, let $\eta^* = \frac{c^*}{4C(M^*+\sqrt{M^*}+1)}$ with $c^* = \frac{\min_{k,k'} c(k,k')}{2} > 0$. Then,

$$\begin{aligned}
&\Pr\left\{B_{ik\tilde{k}} \leq \eta^* C(M^* + \sqrt{M^*} + 1)\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} \leq \eta^* C(M^* + \sqrt{M^*} + 1) - \frac{c^*}{2}\right\} \\
&\qquad + \mathbf{1}\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2 \leq c^*\right\} \\
&\leq \Pr\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} \leq -\frac{c^*}{4}\right\} + \mathbf{1}\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2 \leq c^*\right\}.
\end{aligned}$$

For the first term, use Lemma B5 of Bonhomme and Manresa (2015) again. From Assumption 7-b, we have

$$\Pr\left\{\left|\left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it}\right| \geq z\right\} \leq \Pr\left\{|\dot{U}_{it}| \geq \frac{z}{2M}\right\}.$$

By applying similar argument from before, we can prove the tail property given in Assumption 7-g for $\left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it}$ with any k and $\tilde{k}$. Also, the first part of Assumption 7-g is satisfied since $\left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it}$ is a measurable function of $\left(U_{it}, U_{it-1}\right)$.[4] For any $\nu > 0$,

$$T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)\dot{U}_{it} \leq -\frac{c^*}{4}\right\} = o(1).$$

Again, in the case of Theorem 1, note that E_i has finite support and repeat

$$T_0{}^{\nu} \Pr\left\{\frac{1}{T_0}\sum_t \left(\mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0 = k] - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0 = \tilde{k}]\right)\left(\dot{Y}_{it}(e) - \mathbf{E}[\dot{Y}_{it}(\infty)|k_i^0]\right)\right\} = o(1)$$

for every e. For the second term, Assumption 7-c assumes that $\mathbf{1}\{\frac{1}{T_0}\sum_t \left(\delta_t^0(k) - \delta_t^0(\tilde{k})\right)^2 \leq c^*\} = 0$ when n is large and therefore $o(T^{-\nu})$ for any $\nu > 0$.

Finally, going back to (3) and (5), thanks to K being fixed,

$$\Pr\left\{\hat{k}_i \neq k_i^0, D(\eta^*) = 1\right\} = o(T^{-\nu}). \tag{7}$$

Step 6

In this step let us discuss the probability of assigning a wrong type at least to one unit. As $n \to \infty$, for any $\nu > 0$

$$\begin{aligned}
&\Pr\left\{\sup_i \mathbf{1}_{\{\hat{k}_i \neq k_i^0\}} > 0\right\} \\
&\leq \Pr\left\{\sum_i \mathbf{1}_{\{\hat{k}_i \neq k_i^0\}} > 0, D(\eta^*) = 1\right\} + \Pr\{D(\eta^*) = 0\} \\
&\leq n \cdot \Pr\left\{\hat{k}_i \neq k_i^0, D(\eta^*) = 1\right\} + \Pr\{D(\eta^*) = 0\} \\
&= o(nT_0{}^{-\nu}) + o(1).
\end{aligned}$$

The last equality holds from (7).

□

[4] Here, I am treating $\left\{\delta_t^0(k)\right\}_{t,k}$ as if uniformly fixed across n. This can be relaxed by assuming $\left\{\delta_t^0(k)\right\}_{t,k}$ is also a strongly mixing random process as in Bonhomme and Manresa (2015).

3 Proof for Corollary 3

The first part of the proof is the same with Corollary 2. The second part follows the proof of Theorem 2 of Callaway and Sant'Anna (2021). Fix some t, k and e such that $0 \le e \le t \le T_1 - 1$ and $\mu(k,e) > 0$. Then, it satisfies that $t - e \le \bar{r}_k$ from Assumption 6.

Step 1

Firstly, let us show that $\widehat{CATT}_t(k,e)$ is close to the infeasible estimator using the true types $\{k_i^0\}_{i=1}^n$:

$$\widetilde{CATT}_t(k,e) = \frac{\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\, \mathbf{1}\{k_i^0 = k, E_i = e\}}{\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = e\}} - \frac{\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\, \mathbf{1}\{k_i^0 = k, E_i = \infty\}\pi_e(X_i,k,\hat{\xi})/\pi_\infty(X_i,k,\hat{\xi})}{\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = \infty\}\pi_e(X_i,k,\hat{\xi})/\pi_\infty(X_i,k,\hat{\xi})}.$$

Find that

$$\left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(Y_{it} - Y_{i,e-1}\right) \left(\mathbf{1}\{\hat{k}_i = k, E_i = e\} - \mathbf{1}\{k_i^0 = k, E_i = e\}\right) \frac{\pi_e(X_i,k,\hat{\xi})}{\pi_\infty(X_i,k,\hat{\xi})} \right|$$
$$\le \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n \left(Y_{i,e+r} - Y_{i,e-1}\right)^2 \right)^{\frac{1}{2}} \cdot \left(\frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\hat{k}_i \ne k_i^0\} \right)^{\frac{1}{2}} \sup_i \left| \frac{\pi_e(X_i,k,\hat{\xi})}{\pi_\infty(X_i,k,\hat{\xi})} \right|.$$

$\sup_i \pi_e/\pi_\infty$ is bounded by $1/\varepsilon^\pi$ from Assumption 9-c. $\frac{1}{n}\sum_{i=1}^n \left(Y_{i,e+r} - Y_{i,e-1}\right)^2$ is bounded in expectation uniformly over e and r from Assumption 9-a and therefore $O_p(1)$. From Theorem 2,

$$\Pr\left\{ \sum_{i=1}^n \mathbf{1}\{\hat{k}_i \ne k_i\} > \varepsilon^2 \right\} \le \Pr\left\{ \sup_i \mathbf{1}\{\hat{k}_i \ne k_i\} > 0 \right\} = o(nT_0^{-\nu}) + o(1)$$

for any $\nu, \epsilon > 0$. Since $n{T_0}^{-\nu^*} \to 0$ as $n \to \infty$, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \mathbf{1}\{\hat{k}_i \neq k_i^0\}\right)^{\frac{1}{2}} = o_p(1)$.

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\mathbf{1}\{\hat{k}_i = k, E_i = e\}\frac{\pi_e(X_i, k, \hat{\xi})}{\pi_\infty(X_i, k, \hat{\xi})}\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\mathbf{1}\{k_i^0 = k, E_i = e\}\frac{\pi_e(X_i, k, \hat{\xi})}{\pi_\infty(X_i, k, \hat{\xi})} + o_p(1)\end{aligned}$$

By the same argument,

$$\frac{1}{n}\sum_{i=1}^n \mathbf{1}\{\hat{k}_i = k, E_i = e\}\frac{\pi_e(X_i, k, \hat{\xi})}{\pi_\infty(X_i, k, \hat{\xi})} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = e\}\frac{\pi_e(X_i, k, \hat{\xi})}{\pi_\infty(X_i, k, \hat{\xi})} + o_p(1).$$

The same applies to the other term without π_e/π_∞. Note that $\frac{1}{n}\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = e\}$ and $\frac{1}{n}\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = \infty\}\frac{\pi_e}{\pi_\infty}$ both have nonzero probabilistic limits; for the latter, apply Assumption 9-c and find that it is bounded from below by $\frac{1}{n}\sum_{i=1}^n \mathbf{1}\{k_i^0 = k, E_i = \infty\}\varepsilon^\pi$. Thus,

$$\sqrt{n}\left(\widehat{CATT}_t(k,e) - \widetilde{CATT}_t(k,e)\right) = o_p(1).$$

Step 2

In this step, we rewrite $CATT_t(k,e)$ in a way that it connects to $\widetilde{CATT}_t(k,e)$:

$$CATT_t(k,e) = \mathbf{E}\left[Y_{it}(e) - Y_{i,e-1}(e)|k_i^0 = k, E_i = e\right] - \mathbf{E}\left[Y_{it}(\infty) - Y_{i,e-1}(\infty)|k_i^0 = k, E_i = e\right].$$

Find that

$$\begin{aligned}&\mathbf{E}\left[Y_{it}(\infty) - Y_{i,e-1}(\infty)|k_i^0 = k, E_i = e\right]\\ &= \mathbf{E}\left[\mathbf{E}\left[Y_{it}(\infty) - Y_{i,e-1}(\infty)|X_i, k_i^0 = k\right]|k_i^0 = k, E_i = e\right]\\ &= \mathbf{E}\left[\mathbf{E}\left[Y_{it} - Y_{i,e-1}|X_i, k_i^0 = k, E_i = \infty\right]|k_i^0 = k, E_i = e\right]\\ &= \frac{\mathbf{E}\left[\mathbf{E}\left[Y_{it} - Y_{i,e-1}|X_i, k_i^0 = k, E_i = \infty\right]\mathbf{1}\{k_i^0 = k, E_i = e\}\right]}{\Pr\left\{k_i^0 = k, E_i = e\right\}}\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[Y_{it}-Y_{i,e-1}|X_i,k_i^0=k,E_i=\infty\right]\mathbf{1}\{k_i^0=k,E_i=e\}\right]\\
&=\mathbf{E}\left[\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})\,\mathbf{1}\{k_i^0=k,E_i=\infty\}|X_i\right]\Pr\left\{k_i^0=k,E_i=e|X_i\right\}}{\Pr\left\{k_i^0=k,E_i=\infty|X_i\right\}}\right]\\
&=\mathbf{E}\left[(Y_{it}-Y_{i,e-1})\,\mathbf{1}\{k_i^0=k,E_i=\infty\}\cdot\frac{\Pr\left\{k_i^0=k,E_i=e|X_i\right\}}{\Pr\left\{k_i^0=k,E_i=\infty|X_i\right\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
\Pr\left\{k_i^0=k,E_i=e\right\}&=\mathbf{E}\left[\mathbf{1}\{k_i^0=k,E_i=e\}\cdot\frac{\Pr\left\{k_i^0=k,E_i=\infty|X_i\right\}}{\Pr\left\{k_i^0=k,E_i=\infty|X_i\right\}}\right]\\
&=\mathbf{E}\left[\mathbf{1}\{k_i^0=k,E_i=\infty\}\cdot\frac{\Pr\left\{k_i^0=k,E_i=e|X_i\right\}}{\Pr\left\{k_i^0=k,E_i=\infty|X_i\right\}}\right].
\end{aligned}$$

The second to the last equality holds since $\Pr\left\{E_i=\infty|k_i^0,X_i\right\}\geq\varepsilon^\pi>0$ from Assumption 9-c and $\mu(k,\infty)>0$ for every k from Assumption 6.

For notational brevity, let

$$\begin{aligned}
W_i&=\mathbf{1}\{k_i^0=k,E_i=\infty\}\pi_e(X_i,k,\xi^0)/\pi_\infty(X_i,k,\xi^0),\\
\widehat{W}_i&=\mathbf{1}\{k_i^0=k,E_i=\infty\}\pi_e(X_i,k,\hat{\xi})/\pi_\infty(X_i,k,\hat{\xi}).
\end{aligned}$$

Then,

$$\begin{aligned}
CATT_t(k,e)&=\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})\mathbf{1}\{k_i^0=k,E_i=e\}\right]}{\mathbf{E}\left[\mathbf{1}\{k_i^0=k,E_i=e\}\right]}-\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})W_i\right]}{\mathbf{E}\left[W_i\right]}\\
\widetilde{CATT}_t(k,e)&=\frac{\frac{1}{n}\sum_i(Y_{it}-Y_{i,e-1})\mathbf{1}\{k_i^0=k,E_i=e\}}{\frac{1}{n}\sum_i\mathbf{1}\{k_i^0=k,E_i=e\}}-\frac{\frac{1}{n}\sum_i(Y_{it}-Y_{i,e-1})\widehat{W}_i}{\frac{1}{n}\sum_i\widehat{W}_i}
\end{aligned}$$

Step 3

Now, let us derive an asymptotic linear approximation of $\widetilde{CATT}_t(k,e)$. Find that

$$\sqrt{n}\left(\widetilde{CATT}_t(k,e) - CATT_t(k,e)\right) = A_n - B_n$$

where

$$A_n = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_{it}-Y_{i,e-1})\mathbf{1}\{k_i^0=k,E_i=e\}}{\hat{\mu}(k,e)} - \sqrt{n}\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})\mathbf{1}\{k_i^0=k,E_i=e\}\right]}{\mu(k,e)}$$
$$B_n = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_{it}-Y_{i,e-1})\widehat{W}_i}{\overline{\widehat{W}}_n} - \sqrt{n}\frac{\mathbf{E}\left[(Y_{it}-Y_{i,e-1})W_i\right]}{\mathbf{E}\left[W_i\right]}$$

where $\overline{\widehat{W}}_n = \frac{1}{n}\sum_{i=1}^{n}\widehat{W}_i$.

Before deriving the asymptotic approximation, let us provide some useful expansions and probabilistic convergences. Firstly, apply the first-order Taylor's expansion to $\widehat{W}_i$ with regard to $\hat{\xi}$ around ξ^0:

$$\widehat{W}_i = W_i + \mathbf{1}\{k_i^0=k,E_i=\infty\}\frac{\partial}{\partial\xi}\left(\frac{\pi_e(X_i,k,\xi)}{\pi_\infty(X_i,k,\xi)}\right)^{\intercal}\Bigg|_{\xi\in\left(\xi^0,\hat{\xi}\right)}\left(\hat{\xi}-\xi^0\right). \tag{8}$$

The first-order remainder term is $O_p(1/\sqrt{n})$ since $\|\hat{\xi}-\xi^0\|_2 = O_p(1/\sqrt{n})$ from asymptotic normality of $\hat{\xi}$ and $\frac{\partial}{\partial\xi}\frac{\pi_e}{\pi_\infty} = O_p(1)$ from Assumption 9-d and the convergence of $\hat{\xi}$ to ξ^0:

$$\begin{aligned}\left|\frac{\partial}{\partial\xi}\left(\frac{\pi_e(X_i,k,\xi)}{\pi_\infty(X_i,k,\xi)}\right)^{\intercal}\Bigg|_{\xi\in\left(\xi^0,\hat{\xi}\right)}\left(\hat{\xi}-\xi^0\right)\right| &\le \left\|\frac{\partial}{\partial\xi}\left(\frac{\pi_e(X_i,k,\xi)}{\pi_\infty(X_i,k,\xi)}\right)\Bigg|_{\xi\in\left(\xi^0,\hat{\xi}\right)}\right\|_2\left\|\hat{\xi}-\xi^0\right\|_2 \\ &= O_p(1)O_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Now, apply the second-order Taylor's expansion to $\widehat{W}_i$:

$$\widehat{W}_i = W_i + \mathbf{1}\{k_i^0 = k, E_i = \infty\} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_\infty(X_i, k, \xi)} \right)^\intercal \Bigg|_{\xi = \xi^0} \left(\hat{\xi} - \xi^0 \right)$$
$$+ \mathbf{1}\{k_i^0 = k, E_i = \infty\} \left(\hat{\xi} - \xi^0 \right)^\intercal \frac{\partial^2}{\partial \xi \partial \xi^\intercal} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_\infty(X_i, k, \xi)} \right) \Bigg|_{\xi \in \left(\xi^0, \hat{\xi}\right)} \left(\hat{\xi} - \xi^0 \right). \quad (9)$$

Note that the second-order remainder term is $o_p(1/\sqrt{n})$ from Assumption 9-d and the asymptotic normality of $\hat{\xi}$. An abuse of notation is used when we write $\xi \in (\xi^0, \hat{\xi})$ to say ξ lies between ξ^0 and $\hat{\xi}$. Lastly, find that from (8) and $\frac{1}{n}\sum_i (Y_{it} - Y_{i,e-1})^2$ being bounded in expectation,

$$\left| \frac{1}{n} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \left(\widehat{W}_i - W_i \right) \right| = O_p\left(\frac{1}{\sqrt{n}} \right),$$
$$\frac{1}{n} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \widehat{W}_i = \mathbf{E}\left[(Y_{it} - Y_{i,e-1}) W_i \right] + O_p\left(\frac{1}{\sqrt{n}} \right). \quad (10)$$

The $O_p(1/\sqrt{n})$ term in the second equality comes from applying the CLT to $(Y_{it} - Y_{i,e-1}) W_i$ and the $O_p(1/\sqrt{n})$ term from the first equality. Likewise, we have

$$\overline{\widehat{W}}_n = \mathbf{E}[W_i] + O_p(1/\sqrt{n}). \quad (11)$$

As argued in the Step 1, $\mathbf{E}[W_i] > 0$ from Assumption 9-c.

To drive the asymptotic approximation of B_n, apply the second-order Taylor's expansion to B_n with regard to $\overline{\widehat{W}}_n$ around $\mathbf{E}[W_i]$:

$$\frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \widehat{W}_i}{\overline{\widehat{W}}_n}$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) \widehat{W}_i \left(\frac{1}{\mathbf{E}[W_i]} - \frac{1}{\mathbf{E}[W_i]^2} \left(\overline{\widehat{W}}_n - \mathbf{E}[W_i] \right) + \frac{2}{\widetilde{W}_n^3} \left(\overline{\widehat{W}}_n - \mathbf{E}[W_i] \right)^2 \right)$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{(Y_{it} - Y_{i,e-1}) \widehat{W}_i}{\mathbf{E}[W_i]} - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1}) W_i]}{\mathbf{E}[W_i]^2} \sqrt{n} \left(\overline{\widehat{W}}_n - \mathbf{E}[W_i] \right) + o_p(1).$$

with some $\widetilde{W}_n$ between $\widehat{\overline{W}}_n$ and $\mathbf{E}[W_i]$. The second equality holds from $\mathbf{E}[W_i] > 0$, (11) and (10). Then, from (9) and $\frac{1}{n}\sum_i (Y_{it} - Y_{i,e-1})^2$ being bounded in expectation,

$$\begin{aligned}
&\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n (Y_{it} - Y_{i,e-1})\widehat{W}_i}{\widehat{\overline{W}}_n} \\
&= \frac{1}{\mathbf{E}[W_i]} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_{it} - Y_{i,e-1}) W_i + o_p(1) \\
&\quad + \frac{1}{n} \sum_{i=1}^n \frac{(Y_{it} - Y_{i,e-1}) \mathbf{1}\{k_i^0 = k, E_i = \infty\}}{\mathbf{E}[W_i]} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_\infty(X_i, k, \xi)} \right)^\intercal \Bigg|_{\xi = \xi^0} \cdot \sqrt{n}\left(\hat{\xi} - \xi^0\right) \\
&\quad - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]^2} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^n (W_i - \mathbf{E}[W_i]) \\
&\quad - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]} \frac{1}{n} \sum_{i=1}^n \frac{\mathbf{1}\{k_i^0 = k, E_i = \infty\}}{\mathbf{E}[W_i]} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_\infty(X_i, k, \xi)} \right)^\intercal \Bigg|_{\xi = \xi^0} \cdot \sqrt{n}\left(\hat{\xi} - \xi^0\right).
\end{aligned}$$

Let

$$\begin{aligned}
\bar{B}_1 &= \frac{1}{\mathbf{E}[W_i]} \cdot \mathbf{E}\left[(Y_{it} - Y_{i,e-1})\, \mathbf{1}\{k_i^0 = k, E_i = \infty\} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_\infty(X_i, k, \xi)} \right) \Bigg|_{\xi = \xi^0} \right] \\
\bar{B}_2 &= \frac{1}{\mathbf{E}[W_i]} \cdot \mathbf{E}\left[\mathbf{1}\{k_i^0 = k, E_i = \infty\} \frac{\partial}{\partial \xi} \left(\frac{\pi_e(X_i, k, \xi)}{\pi_\infty(X_i, k, \xi)} \right) \Bigg|_{\xi = \xi^0} \right].
\end{aligned}$$

Note that the sample analogues for $\bar{B}_1$ and $\bar{B}_2$ with ξ^0 replaced with $\hat{\xi}$ are consistent for $\bar{B}_1$ and $\bar{B}_2$ from Assumption 9-d. Consequently,

$$\begin{aligned}
B_n = &\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{W_i}{\mathbf{E}[W_i]} \left(Y_{it} - Y_{i,e-1} - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]} \right) \\
&+ \left(\bar{B}_1 - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})W_i]}{\mathbf{E}[W_i]} \bar{B}_2 \right)^\intercal \cdot \sqrt{n}\left(\hat{\xi} - \xi^0\right) + o_p(1).
\end{aligned}$$

By repeating the same argument for A_n,

$$A_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\mathbf{1}\{k_i^0 = k, E_i = e\}}{\mu(k, e)} \left(Y_{it} - Y_{i,e-1} - \frac{\mathbf{E}[(Y_{it} - Y_{i,e-1})\mathbf{1}\{k_i^0 = k, E_i = e\}]}{\mu(k, e)} \right) + o_p(1).$$

Note the asymptotic linear approximation given in Corollary 3 holds for $\hat{\xi}$ as well from the proof for Corollary 2. We can construct score functions l^1 and l^0 as follows:

$$\begin{aligned}
A_n &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^1_{tke}\left(\{Y_{it}\}_{t \geq -1}, k_i^0, E_i\right) + o_p(1), \\
B_n &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^0_{tke}\left(\{Y_{it}\}_{t \geq -1}, X_i, k_i^0, E_i\right) + o_p(1).
\end{aligned}$$

Note that l^π appears in l^0. Now we have

$$\begin{aligned}
&\sqrt{n}\left(\widehat{CATT}_t(k,e) - CATT_t(k,e)\right) \\
&= \begin{pmatrix} 1, -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^1_{tke}\left(\{Y_{it}\}_{t \geq -1}, k_i^0, E_i\right) \\ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l^0_{tke}\left(\{Y_{it}\}_{t \geq -1}, X_i, k_i^0, E_i\right) \end{pmatrix} + o_p(1).
\end{aligned}$$

The asymptotic linear approximation is derived for $\widehat{CATT}_t(k,e)$.

Step 4

To derive asymptotic distribution of $\hat{\beta}_r(k)$, consider

$$\begin{aligned}
&\frac{\hat{\mu}(k,e)}{\sum_{e' \leq T_1 - 1 - r} \hat{\mu}(k,e')} \cdot \sqrt{n}\widehat{CATT}_t(k,e) - \frac{\mu(k,e)}{\sum_{e' \leq T_1 - 1 - r} \mu(k,e')} \cdot \sqrt{n} CATT_t(k,e) \\
&= \frac{\hat{\mu}(k,e)}{\sum_{e' \leq T_1 - 1 - r} \hat{\mu}(k,e')} \cdot \sqrt{n}\left(\widehat{CATT}_t(k,e) - CATT_t(k,e)\right) \\
&\quad + \sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e' \leq T_1 - 1 - r} \hat{\mu}(k,e')} - \frac{\mu(k,e)}{\sum_{e' \leq T_1 - 1 - r} \mu(k,e')}\right) \cdot CATT_t(k,e).
\end{aligned}$$

By taking the second-order Taylor's expansion of $\sum_{e'} \hat{\mu}(k,e')$ around $\sum_{e'} \mu(k,e')$,

$$\begin{aligned}
\sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')} - \frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right) = \sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\mu(k,e')} - \frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right) \\
- \frac{\hat{\mu}(k,e)}{(\sum_{e'}\mu(k,e'))^2}\sqrt{n}\left(\sum_{e'}\Big(\hat{\mu}(k,e') - \mu(k,e')\Big)\right) \\
+ \frac{2\hat{\mu}(k,e)}{\tilde{\mu}^3}\sqrt{n}\left(\sum_{e'}\Big(\hat{\mu}(k,e') - \mu(k,e')\Big)\right)^2
\end{aligned}$$

with some $\tilde{\mu}$ between $\sum_{e'} \mu(k,e')$ and $\sum_{e'} \hat{\mu}(k,e')$. The second-order remainder term is $o_p(1)$ since $\sqrt{n}\left(\sum_{e'}(\hat{\mu}(k,e') - \mu(k,e'))\right) = O_p(1)$ and $\sum_{e'} \mu(k,e')$ is nonzero by taking $r \leq \bar{r}_k$ from Assumption 6. Thus,

$$\begin{aligned}
&\sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')} - \frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right) \\
&= \sqrt{n}\left(\frac{\hat{\mu}(k,e) - \mu(k,e)}{\sum_{e'}\mu(k,e')}\right) - \frac{\mu(k,e)}{(\sum_{e'}\mu(k,e'))^2}\sqrt{n}\left(\sum_{e'}\Big(\hat{\mu}(k,e') - \mu(k,e')\Big)\right) + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbf{1}\{k_i^0 = k, E_i = e\} - \mu(k,e)}{\sum_{e'}\mu(k,e')} \\
&\quad - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mu(k,e)\big(\mathbf{1}\{k_i^0 = k, E_i \leq T_1 - 1 - r\} - \sum_{e'}\mu(k,e')\big)}{\big(\sum_{e'}\mu(k,e')\big)^2} + o_p(1).
\end{aligned}$$

Let l^{μ} denote the score function in the asymptotic linear approximation:

$$\sqrt{n}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')} - \frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} l^{\mu}_{ke}(k_i^0, E_i) + o_p(1).$$

Combining all of the results so far, we get

$$
\begin{aligned}
&\sqrt{n}\left(\hat{\beta}_r(k)-\beta_r(k)\right)\\
&=\sum_{e\leq T_1-1-r}\left(\frac{\hat{\mu}(k,e)}{\sum_{e'}\hat{\mu}(k,e')}\cdot\sqrt{n}\widehat{CATT}_t(k,e)-\frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\cdot\sqrt{n}CATT_t(k,e)\right)\\
&=\sum_{e\leq T_1-1-r}\frac{\mu(k,e)}{\sum_{e'}\mu(k,e')}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(l^1_{e+r,k,e}\left(\{Y_{it}\}_{t\geq 0},k_i^0,E_i\right)-l^0_{e+r,k,e}\left(\{Y_{it}\}_{t\geq 0},X_i,k_i^0,E_i\right)\right)\\
&\quad+\sum_{e\leq T_1-1-r}CATT_t(k,e)\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}l^{\mu}_{ke}(k_i^0,E_i)+o_p(1).
\end{aligned}
$$

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